

A NOTE ON HERMITE-FEJÉR INTERPOLATION ON THE NON-UNIFORM NODES OF THE UNIT CIRCLE

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Abstract: This paper deals with the Hermite-Fejér interpolation problem on the unit circle with the nodal system containing the vertically projected zeros of Jacobi's polynomial with boundary points on the unit circle. We worked upon three nodal structures throughout this paper and obtained rate of convergence for each case. Moreover, we did a comparison of all the three cases and provided some important conclusions.

Keywords and Phrases: Interpolation, non-uniform nodes, Jacobi Polynomial, Rate of Convergence, Hermite-Fejér.

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1. Introduction

The study of Hermite-Fejér interpolation has drawn interest of many researchers in past years and has been a subject of investigation for a quite long time. Earlier in 1974, [Saxena 12] showed Hermite-Fejér interpolation process for the function on the nodes given in the paper of [Berman 3], converged uniformly to function in $[-1, 1]$. [Daruís and Gonzalez-Vera 9] in 2001 extended Fejér's classical result onto the unit circle and studied the convergence of Hermite-Fejér interpolation polynomial by using Laurent polynomial.

After a decade, in 2011 [Berriochoa et al. 4] came out with some improvements to the Hermite-Fejér interpolation on the unit circle. In another paper, [Berriochoa et al. 5] dealt with the order of convergence of the Laurent polynomial of

Hermite-Fejér interpolation on the unit circle. Supremum norm of the error was not only obtained for analytic functions but also for their corresponding asymptotic constants.

Series of recent research works by [Berriochoa et al. 6, 7, 8], [Mastroianni et al. 11], [Bahadur & Varun 1, 2] and [Mathur & Mathur 10] motivated us to consider a problem of Hermite-Fejér interpolation for the non-uniform nodes on the unit circle. Let $Z_n = \{z_k; k = 0(1)2n + 1\}$ satisfying

$$\left\{ z_0 = 1, z_{2n+1} = -1, z_k = \cos\theta_k + i\sin\theta_k, z_{n+k} = -z_k, k = 1(1)2n \right\},$$

be the vertical projections of zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$, where $x_k = \cos\theta_k$, $k = 1(1)n$ are zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n such that

$$1 > x_1 > x_2 > \dots > x_n > -1.$$

In this paper, we took three cases to study our problem.

Case I : Interpolatory polynomial $F_n(z)$ of degree $\leq 4n - 1$ satisfying following conditions.

$$\begin{cases} F_n(f, z_k) = \alpha_k, & k = 1(1)2n, \\ F'_n(f, z_k) = 0, & k = 1(1)2n. \end{cases} \quad (1.1)$$

Case II : Interpolatory polynomial $G_n(z)$ of degree $\leq 4n + 1$ satisfying following conditions.

$$\begin{cases} G_n(f, z_k) = \beta_k, & k = 0(1)2n + 1, \\ G'_n(f, z_k) = 0, & k = 1(1)2n. \end{cases} \quad (1.2)$$

Case III : Interpolatory polynomial $H_n(z)$ of degree $\leq 4n + 3$ satisfying following conditions.

$$\begin{cases} H_n(f, z_k) = \gamma_k, & k = 0(1)2n + 1, \\ H'_n(f, z_k) = 0, & k = 0(1)2n + 1. \end{cases} \quad (1.3)$$

The paper has been organised in a following manner. Section 2 is assigned to preliminaries. In section 3, we give explicit formulae for the interpolatory polynomial. Section 4 is devoted to the estimates, which comes in handy to determine the rate of convergence of the interpolatory polynomials covered in section 5. To sum up, we have included a conclusion section to view the valuable aspect of this research.

2. Preliminaries

In this section, we provide the following well known results, which we shall use.

$$W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n P_n^{(\alpha,\beta)} \left(\frac{1+z^2}{2z} \right) z^n, \quad (2.1)$$

$$K_n = 2^{2n} n! \frac{\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)}, \quad (2.2)$$

$$R(z) = (z^2 - 1)W(z). \quad (2.3)$$

The differential equation satisfied by $P_n^{(\alpha, \beta)}(x)$ is

$$(1 - x^2)P_n^{(\alpha, \beta)''}(x) + [\beta - \alpha - (\alpha + \beta + 2)x]P_n^{(\alpha, \beta)'}(x) + n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) = 0, \quad (2.4)$$

where $x = \frac{1 + z^2}{2z}$.

The fundamental polynomials of Lagrange interpolation on the zeros of $W(z)$ and $R(z)$ is respectively given by (2.5) and (2.6)

$$L_{1k}(z) = \frac{W(z)}{(z - z_k)W'(z_k)}, \quad k = 1(1)2n \quad (2.5)$$

$$L_k(z) = \frac{R(z)}{(z - z_k)R'(z_k)}, \quad k = 0(1)2n + 1 \quad (2.6)$$

For $-1 \leq x \leq 1$,

$$(1 - x^2)^{1/2} |P_n^{(\alpha, \beta)}(x)| = O(n^{\alpha-1}), \quad (2.7)$$

$$|P_n^{(\alpha, \beta)}(x)| = O(n^\alpha), \quad (2.8)$$

$$(1 - x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \quad (2.9)$$

$$|P_n^{(\alpha, \beta)'}(x_k)| \sim k^{-\alpha - \frac{3}{2}} n^{\alpha+2}. \quad (2.10)$$

For more details, refer [Szegő 13].

3. Explicit Representation of Interpolatory Polynomial

Referring to the Case I, II and III mentioned in (1.1) - (1.3), we give following theorems.

Theorem 3.1. *Let $F_n(z)$ be a polynomial satisfying (1.1), given by*

$$F_n(z) = \sum_{k=1}^{2n} \alpha_k A_k(z), \quad (3.1)$$

where $A_k(z)$ is a unique polynomial of degree $\leq 4n - 1$, such that

$$A_k(z) = L_{1k}^2(z) \left\{ 1 - 2(z - z_k) L'_{1k}(z_k) \right\}. \quad (3.2)$$

satisfying the conditions

$$\begin{cases} A_k(z_j) = \delta_{kj}, & j, k = 1(1)2n, \\ A'_k(z_j) = 0, & j, k = 1(1)2n. \end{cases} \quad (3.3)$$

Theorem 3.2. Let $G_n(z)$ be a polynomial satisfying (1.2), given by

$$G_n(z) = \sum_{k=0}^{2n+1} \beta_k B_k(z), \quad (3.4)$$

where $B_k(z)$ is a unique polynomial of degree $\leq 4n + 1$, such that

For $k = 1(1)2n$

$$B_k(z) = - \left(\frac{L'_{1k}(z_k) + L'_k(z_k)}{W'(z_k)} \right) W(z) L_k(z) + L_{1k}(z) L_k(z), \quad (3.5)$$

For $k = 0, 2n + 1$

$$B_k(z) = \frac{W(z) L_k(z)}{W(z_k)}, \quad (3.6)$$

satisfying the conditions

$$\begin{cases} B_k(z_j) = \delta_{kj}, & j, k = 0(1)2n + 1, \\ B'_k(z_j) = 0, & j = 1(1)2n, k = 0(1)2n + 1. \end{cases} \quad (3.7)$$

Theorem 3.3. Let $H_n(z)$ be a polynomial satisfying (1.3), given by

$$H_n(z) = \sum_{k=0}^{2n+1} \gamma_k C_k(z), \quad (3.8)$$

where $C_k(z)$ is a unique polynomial of degree $\leq 4n + 3$, such that

$$C_k(z) = L_k^2(z) \left\{ 1 - 2(z - z_k) L'_k(z_k) \right\}, \quad (3.9)$$

satisfying the conditions

$$\begin{cases} C_k(z_j) = \delta_{kj}, & j, k = 0(1)2n + 1, \\ C'_k(z_j) = 0, & j, k = 0(1)2n + 1. \end{cases} \quad (3.10)$$

Proof of theorem 3.1. Let

$$A_k(z) = a_k W(z) L_{1k}(z) + L_{1k}^2(z), \quad (3.11)$$

where $A_k(z)$ is atmost of degree $4n - 1$ satisfying the conditions given in (3.3), provides us with

$$a_k = \frac{-2L'_{1k}(z_k)}{W'(z_k)}. \quad (3.12)$$

Substituting (2.1), (2.5) and (3.12) in (3.11), the theorem follows.

Proof of theorem 3.2. For $k = 1(1)2n$, Let

$$B_k(z) = b_k W(z) L_k(z) + L_{1k}(z) L_k(z), \quad (3.13)$$

Using conditions given in (3.7), we get

$$b_k = -\left(\frac{L'_{1k}(z_k) + L'_k(z_k)}{W'(z_k)}\right), \quad (3.14)$$

For $k = 0, 2n + 1$

Let

$$B_k(z) = b_{1k} W(z) L_k(z), \quad (3.15)$$

Using (3.7), we have

$$b_{1k} = \frac{1}{W(z_k)}. \quad (3.16)$$

From (3.13), (3.14), (3.15) and (3.16), the theorem follows.

Proof of theorem 3.3. Proceed similarly as in proof of *Theorem 3.1*.

4. Estimates of Fundamental Polynomials

Lemma 4.1. Let $A_k(z)$ be given by (3.2), then

$$\sum_{k=1}^{2n} |A_k(z)| \leq \begin{cases} C \log n, & -1 < \alpha \leq -\frac{1}{2}, \\ Cn^{\frac{1}{2}+\alpha} \log n, & -\frac{1}{2} < \alpha \leq \frac{1}{2}, \\ Cn^{\frac{1}{2}+\alpha}, & \alpha > \frac{1}{2}, \end{cases} \quad (4.1)$$

where C is a constant independent of n and z .

Lemma 4.2. Let $B_k(z)$ be given by (3.5) and (3.6), then

$$\sum_{k=0}^{2n+1} |B_k(z)| \leq \begin{cases} C \log n, & -1 < \alpha \leq 0, \\ Cn^{2\alpha} \log n, & 0 < \alpha \leq \frac{1}{2}, \\ Cn^{2\alpha}, & \alpha > \frac{1}{2}, \end{cases} \quad (4.2)$$

where C is a constant independent of n and z .

Lemma 4.3. Let $C_k(z)$ be given by (3.9), then

$$\sum_{k=0}^{2n+1} |C_k(z)| \leq \begin{cases} C \log n, & -1 < \alpha \leq \frac{1}{2}, \\ Cn^{2\alpha-1} \log n, & \frac{1}{2} < \alpha \leq 1, \\ Cn^{2\alpha-1}, & \alpha > 1, \end{cases} \quad (4.3)$$

where C is a constant independent of n and z .

Proof of the lemma 4.1.

$$|A_k(z)| = \left| L_{1k}^2(z) - \frac{2L'_{1k}(z_k)L_{1k}(z)W(z)}{W'(z_k)} \right|, \quad (4.4)$$

$$\sum_{k=1}^{2n} |A_k(z)| \leq \max |L_{1k}(z)| \sum_{k=1}^{2n} |L_{1k}(z)| + 2 \sum_{k=1}^{2n} \frac{|L'_{1k}(z_k)| |L_{1k}(z)| |W(z)|}{|W'(z_k)|}, \quad (4.5)$$

Using (2.5), (2.7), (2.9) and (2.10), we have

$$\sum_{k=1}^{2n} |A_k(z)| \leq C_1 \sum_{k=1}^{2n} \frac{1}{k^{-\alpha+\frac{1}{2}}} + C_2 \sum_{k=1}^{2n} \frac{1}{k^{-2\alpha+1}}, \quad (4.6)$$

where C_1 and C_2 are constants independent of n and z . Further calculations give our desired lemma.

Proof of the lemma 4.2. Let

$$\sum_{k=0}^{2n+1} |B_k(z)| \leq I_1 + I_2, \quad (4.7)$$

where

$$I_1 \leq \sum_{k=1}^{2n} \frac{|L'_{1k}(z_k)| |L_k(z)| |W(z)|}{|W'(z_k)|} + \sum_{k=1}^{2n} \frac{|L'_k(z_k)| |L_k(z)| |W(z)|}{|W'(z_k)|} \quad (4.8)$$

$$+ \sum_{k=1}^{2n} |L_{1k}(z)| |L_k(z)|. \quad (4.9)$$

Using (2.1), (2.5), (2.6), (2.7), (2.9) and (2.10), we have

$$I_1 \leq C \log n \quad (-1 < \alpha \leq 0), \quad (4.10)$$

$$I_2 \leq \sum_{k=0, 2n+1} \frac{|L_k(z)| |W(z)|}{|W(z_k)|}, \quad (4.11)$$

Using (2.2), (2.6), (2.8) and (2.10), we have

$$I_2 \leq C \quad (-1 < \alpha \leq 0), \quad (4.12)$$

Combining (4.9) and (4.11), after a little computations for different ranges of α , the lemma follows.

Proof of the lemma 4.3. Proceed similarly as in proof of Lemma 4.1.

5. Convergence

Remark: Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$ and $f^{(r)} \in Lip\,q$, $q > 0$, then the sequence $\{F_n(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, which follows from (5.5) as

$$\omega_r(f, n^{-1}) = O(n^{-r+1-q}) \quad \{q > \frac{3}{2} + \alpha - r\}, \quad (5.1)$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$, such that

$$\left[r - \frac{3}{2} \leq \alpha < r - \frac{1}{2}\right], \text{ where } r = 2, 3, \dots \quad (5.2)$$

To prove the Theorem 5.1, we shall need following.

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then, there exists a polynomial $L_n(z)$ of degree $\leq 4n - 1$ satisfying Jackson's inequality.

$$|f(z) - L_n(z)| = \begin{cases} C\omega(f, n^{-1}), & -1 < \alpha \leq -\frac{1}{2}, \\ C\omega_2(f, n^{-1}), & -\frac{1}{2} < \alpha \leq \frac{1}{2}, \\ C\omega_r(f, n^{-1}), & \alpha > \frac{1}{2}, \end{cases} \quad (5.3)$$

where C is a constant independent of n and z .

Theorem 5.1. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then $F_n(z)$ defined by

$$F_n(z) = \sum_{k=1}^{2n} f(z_k) A_k(z). \quad (5.4)$$

satisfies the relation

$$|F_n(z) - f(z)| = \begin{cases} O(\omega(f, n^{-1}) \log n); & -1 < \alpha \leq -\frac{1}{2}, \\ O(\omega_2(f, n^{-1}) \log n); & -\frac{1}{2} < \alpha \leq \frac{1}{2}, \\ O(\omega_r(f, n^{-1}) n^{\frac{1}{2}+\alpha}); & \alpha > \frac{1}{2}, \end{cases} \quad (5.5)$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$.

Proof. Since $F_n(z)$ be the uniquely determined polynomial of degree $\leq 4n-1$ and the polynomial $L_n(z)$ satisfying (5.3) can be expressed as

$$L_n(z) = \sum_{k=1}^{2n} L_n(z_k) A_k(z), \quad (5.6)$$

Then

$$|F_n(z) - f(z)| \leq |F_n(z) - L_n(z)| + |L_n(z) - f(z)|, \quad (5.7)$$

$$|F_n(z) - f(z)| \leq \sum_{k=1}^{2n} |f(z_k) - L_n(z_k)| |A_k(z)| + |L_n(z) - f(z)|. \quad (5.8)$$

Using equations (5.3), (5.4) and Lemma 4.1, we have Theorem 5.1.

Theorem 5.2. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then $G_n(z)$ defined by

$$G_n(z) = \sum_{k=0}^{2n+1} f(z_k) B_k(z), \quad (5.9)$$

satisfies the relation

$$|G_n(z) - f(z)| = \begin{cases} O(\omega(f, n^{-1}) \log n), & -1 < \alpha \leq 0, \\ O(\omega_2(f, n^{-1}) \log n), & 0 < \alpha \leq \frac{1}{2}, \\ O(\omega_r(f, n^{-1}) n^{2\alpha}), & \alpha > \frac{1}{2}, \end{cases} \quad (5.10)$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$.

Theorem 5.3. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then $H_n(z)$ defined by

$$H_n(z) = \sum_{k=0}^{2n+1} f(z_k) C_k(z), \quad (5.11)$$

satisfies the relation

$$|H_n(z) - f(z)| = \begin{cases} O(\omega(f, n^{-1}) \log n), & -1 < \alpha \leq \frac{1}{2}, \\ O(\omega_2(f, n^{-1}) \log n), & \frac{1}{2} < \alpha \leq 1, \\ O(\omega_r(f, n^{-1}) n^{2\alpha-1}), & \alpha > 1, \end{cases} \quad (5.12)$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$.

Proof of Theorem 5.2 and Theorem 5.3 can be obtained proceeding similarly as in Theorem 5.1.

6. Conclusion

This paper contains a problem involving three nodal structures different from each other in a perspective of inclusion of the end points(± 1) on the real line. We can conclude from all the calculations done in this paper that increase in nodal points gradually increase the degree of the interpolatory polynomial formed, but the convergence came out to be more stronger in Case I. Thus, for certain modulus of continuity, the interpolatory polynomial in the Case I, where function values and the derivative values are prescribed on Z_n excluding end points ± 1 , can survive through high range of α for convergence purpose as compared to interpolatory polynomials obtained for Case II and Case III.

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